Asymptotic Log-Harnack inequality and ergodicity for path-dependent SDEs with infinite memory

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Outline

Introduction

In the first place, we focus on a path-dependent SDE on $\mathbb{R}^n$

$$dX(t) = b(X_t)dt + \sigma(X_t)dW(t), \quad t > 0, \quad X_0 = \xi \in \mathcal{C}_r, \quad (2.1)$$

where, for each fixed $t \geq 0$, $X_t(\cdot) \in \mathcal{C}_r$, which is defined as $X_t(\theta) = X(t+\theta), -\infty < \theta \leq 0$, $b : \mathcal{C}_r \to \mathbb{R}^n$, $\sigma : \mathcal{C}_r \to \mathbb{R}^n \otimes \mathbb{R}^n$, and $(W(t))_{t\geq 0}$ is an n-dimensional Brownian motion defined on some complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.
Let $\mathcal{C} = C((-\infty, 0]; \mathbb{R}^n)$ denote the family of all continuous mappings $f : (-\infty, 0] \to \mathbb{R}$. For a fixed constant $r > 0$, set

$$\mathscr{C}_r := \left\{ \phi \in \mathcal{C} : \|\phi\|_r := \sup_{-\infty<\theta\leq 0} (\mathrm{e}^{r\theta}|\phi(\theta)|) < \infty \right\}. \qquad (2.2)$$

It is a Polish (i.e., complete, separable, metrizable) space with the norm $\|\cdot\|_r$.

Let $\mathcal{M}_0 = \mathcal{M}_0((-\infty, 0])$ stand for the collection of all probability measures on $(-\infty, 0]$. For $\kappa > 0$, set

$$\mathcal{M}_\kappa := \Big\{\mu \in \mathcal{M}_0 : \mu^{(\kappa)} := \int_{-\infty}^{0} \mathrm{e}^{-\kappa\theta}\mu(d\theta) < \infty\Big\}.$$

Denote $\mathcal{B}_b(\mathcal{C}_r)$ by the family of all bounded measurable functions $\phi : \mathcal{C}_r \to \mathbb{R}$ with the uniform norm $\|\phi\|_\infty := \sup_{\xi\in\mathcal{C}_r} |\phi(\xi)|$. For a Polish space $\mathcal{X}$, let $\mathcal{P}(\mathcal{X})$ be the family of all probability measures on $\mathcal{X}$. For $\mu_1, \mu_2 \in \mathcal{P}(\mathcal{X})$, let $\mathcal{C}(\mu_1, \mu_2)$ be the set of all probability measures on $\mathcal{X}^2 := \mathcal{X} \times \mathcal{X}$ with marginals μ_1 and μ_2, respectively.

For the drift term $b(\cdot)$ and the diffusion term $\sigma(\cdot)$, we assume that

(H1) $b(\cdot)$ is continuous and bounded on bounded subsets of $\mathcal{C}_r$ and there exists $K > 0$ such that

$$2\langle \xi(0)-\eta(0), b(\xi)-b(\eta)\rangle^{+}+\|\sigma(\xi)-\sigma(\eta)\|_{\mathrm{HS}}^{2} \leq K\|\xi-\eta\|_{r}^{2},\ \xi,\eta\in\mathcal{C}_r; \tag{2.3}$$

(H2) $\sup_{\xi\in\mathcal{C}_r}\|\sigma(\xi)\| < \infty$, and, for each $\xi \in \mathcal{C}_r$, $\sigma(\xi)$ is invertible with $\sup_{\xi\in\mathcal{C}_r}\|\sigma^{-1}(\xi)\| < \infty$.

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Under (**H1**), (2.1) has a unique strong solution $(X^{\xi}(t))_{t\geq 0}$ with the initial value $X_0=\xi$ by following the argument of [Renesse and Scheutzow], see also [Wu, Yin and Mei]. Generally speaking, X_t^{ξ} is called the segment process (or functional solution) corresponding to the solution process $X^{\xi}(t)$. Moreover, $(X_t^{\xi})_{t\geq 0}$ is a homogeneous strong Markov process. Let $\mathcal{P}_t$ be Markov semigroup generated by X_t^{ξ}, i.e., $\mathcal{P}_t f(\xi)=\mathbb{E}f(X_t^{\xi}), f\in\mathcal{B}_b(\mathcal{C}_r)$. It is worth pointing out that the Markov semigroup $\mathcal{P}_t$ generated by the segment process X_t^{ξ} does not enjoy the strong Feller property, in particular, in the case that the diffusion term $\sigma(\cdot)$ depends on the past. For more details, please refer to [Hairer, Mattingly and Scheutzow, 2009].

A pseudo-metric for a Polish space $\mathcal{X}$ is a continuous function $d : \mathcal{X}^2 \to \mathbb{R}_+ := [0, \infty)$ such that $d(x, x) = 0$ and $d(x, y) \le d(x, z) + d(z, y)$ for $x, y, z \in \mathcal{X}$. For an increasing sequence of pseudo-metrics $\{d_n\}_{n=0}^{\infty}$ (i.e., $d_i(\cdot, \cdot) \ge d_j(\cdot, \cdot), i \ge j$) on the Polish space $\mathcal{X}$, if $\lim_{n\to\infty} d_n(x, y) = 1$ for all $x \ne y$, then $\{d_n\}$ is called a totally separating system of pseudo-metrics for $\mathcal{X}$.

Definition

A Markov transition semigroup $\mathcal{P}_t$ on a Polish space $\mathcal{X}$ is called asymptotically strong Feller at $\xi \in \mathcal{X}$ if there exist a totally separating system of pseudo-metrics $\{d_n\}$ for $\mathcal{X}$ and a sequence t_k such that

$$\inf_{U \in \mathcal{U}_\xi} \limsup_{k \to \infty} \sup_{\eta \in U} d_k(\mathcal{P}_{t_k}(\xi, \cdot), \mathcal{P}_{t_k}(\eta, \cdot)) = 0, \tag{2.4}$$

where $\mathcal{U}_\xi$ denotes the collection of all open sets containing ξ and

$$d_k(\mathcal{P}_{t_k}(\xi, \cdot), \mathcal{P}_{t_k}(\eta, \cdot)) := \inf_{\mu \in \mathcal{C}(\mathcal{P}_{t_k}(\xi, \cdot), \mathcal{P}_{t_k}(\eta, \cdot))} \int_{\mathcal{X}^2} d_k(x, y) \mu(dx, dy).$$

$\mathcal{P}_t$ is called asymptotically strong Feller if (2.4) holds at every $\xi \in \mathcal{X}$.

Definition

A Markov semigroup $(\mathcal{P}_t)_{t\geq 0}$ on a Polish space $(\mathcal{X}, d)$ is said to satisfy the asymptotic log-Harnack inequality if there exist $\Phi \in C(\mathcal{X}^2; \mathbb{R}_+)$ and $\Psi \in C(\mathbb{R}_+ \times \mathcal{X}^2; \mathbb{R}_+)$ such that

$$\begin{aligned} \mathcal{P}_t \log f(x) \leq \log \mathcal{P}_t f(y) + \Phi(x, y) + \Psi(t, x, y) \|\nabla \log f\|_\infty, \\ x, y \in \mathcal{X}, f \in C_b^1(\mathcal{X}; \mathbb{R}_+), \end{aligned} \tag{2.5}$$

where, for any $x, y \in \mathcal{X}$, $\Phi(x, y) = \Phi(y, x)$, $\Psi(t, x, y) = \Psi(t, y, x)$, $\Phi(x, x) = \Psi(t, x, x) = 0$, and $\lim_{t\to\infty} \sup_{\{y:d(x,y)\leq\delta\}} \Psi(t, x, y) = 0$ for some $\delta > 0$.

Theorem

Under (**H1**) and (**H2**), there exist $C_1, C_2 > 0$ such that

$$\mathcal{P}_t \log f(\eta) \leq \log \mathcal{P}_t f(\xi) + C_1 \|\xi - \eta\|_r^2 + C_2\, \mathrm{e}^{-rt} \|\nabla \log f\|_\infty \|\xi - \eta\|_r \tag{2.6}$$

holds for $\xi, \eta \in \mathcal{C}_r$ and $f \in C_b^1(\mathcal{X}; \mathbb{R}_+)$. Moreover, (5.2) implies that $\mathcal{P}_t$ is asymptotically strong Feller.

Theorem

Let P_t *satisfy* (2.6) *for some symmetric* $\Phi, \Psi_t : E \times E \to \mathbb{R}_+$ *with* $\Psi_t \downarrow 0$ *as* $t \uparrow \infty$. *Then:*

(1) **(Gradient estimate)** *If, for any* $x \in E$,

$$\Lambda(x) := \limsup_{y \to x} \frac{\Phi(x,y)}{\rho(x,y)^2} < \infty, \quad \text{and } \Gamma_t(x) := \limsup_{y \to x} \frac{\Psi_t(x,y)}{\rho(x,y)} < \infty, \tag{2.7}$$

then, for any $t > 0$ *and* $f \in \mathrm{Lip}_b(E) := \mathrm{Lip}(E) \cap \mathcal{B}_b(E)$,

$$|\nabla P_t f| \le 2\Lambda P_t f^2 - (P_t f)^2 + \|\nabla f\|_\infty \Gamma_t. \tag{2.8}$$

In particular, when $\Gamma_t \downarrow 0$ *as* $t \uparrow \infty$, P_t *is asymptotically strong Feller.*

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(2) **(Asymptotic heat kernel estimate)** *If P_t has an invariant probability measure μ, then, for any $f \in \mathcal{B}_b^+(E)$ with $\|\nabla f\|_\infty < \infty$,*

$$\limsup_{t\to\infty} P_t f(x) \le \log\left(\frac{\mu(e^f)}{\int_E e^{-\Phi(x,y)}\mu(y)}\right), \quad x \in E. \tag{2.9}$$

Consequently, for any closed $A \subset E$ with $\mu(A) = 0$,

$$\limsup_{t\to\infty} P_t 1_A(x) = 0, \quad x \in E. \tag{2.10}$$

(3) **(Uniqueness of invariant probability)** *P_t has at most one invariant probability measure.*

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(4) **(Asymptotic irreducibility)** *Let $x \in E$ and measurable set $A \subset E$ such that*

$$\delta(x, A) := \liminf_{t\to\infty} P_t(x, A) > 0.$$

Then

$$\liminf_{t\to\infty} P_t(y, A_\varepsilon) > 0, \quad y \in E, \varepsilon > 0. \qquad (2.11)$$

Moreover, for any $\varepsilon_0 \in (0, \delta(x, A))$, there exists a constant $t_0 > 0$ such that

$$P_t(y, A_\varepsilon) > 0 \ \text{ provided } \ t \geq t_0, \ \ \Psi_t(x, y) < \varepsilon\varepsilon_0. \qquad (2.12)$$

For a measurable set $A \subset E$ and $x \in E$, let $\rho(x, A) = \inf_{y\in A} \rho(x, y)$ be the distance between x and A. Moreover, for any $\varepsilon > 0$, let $A_\varepsilon = \{y \in E : \rho(y, A) < \varepsilon\}$.

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Sketch of the Proof

We shall adopt the asymptotic coupling approach. For any $\lambda > r$, where $r > 0$ is the constant introduced in (2.2), let us consider a path-dependent SDE

$$dY(t) = \{b(Y_t) + \lambda\sigma(Y_t)\sigma^{-1}(X_t)(X(t) - Y(t))\}dt + \sigma(Y_t)dW(t), \quad t > 0,\ Y_0 = \eta \in \mathcal{C}_r, \tag{2.13}$$

Under (**H1**) and (**H2**), (2.13) has a unique strong solution $(Y^\eta(t))_{t\geq 0}$ with the initial value $Y_0 = \eta$, where the corresponding segment process is written as $(Y_t^\eta)_{t\geq 0}$.

For any $t \geq 0$ and $\lambda > r$, set

$$h_1^{(\lambda)}(t) := \lambda\, \sigma^{-1}(X_t^{\xi})(X^{\xi}(t) - Y^{\eta}(t)),$$
$$\tilde{W}(t) := W(t) + \int_0^t h_1^{(\lambda)}(s)ds.$$

Define

$$R_{\infty}^{(1)} = \exp\Big(- \int_0^{\infty} \langle h_1^{(\lambda)}(t), dW(t)\rangle - \frac{1}{2}\int_0^{\infty} |h_1^{(\lambda)}(t)|^2 dt\Big).$$

Then, according to the Girsanov theorem, $(\tilde{W}(t))_{t\geq 0}$ is a Brownian motion under the weighted probability measure $d\mathbb{Q} := R_{\infty}^{(1)} d\mathbb{P}$.

Lemma

Assume that (**H1**) and (**H2**) hold. Then there exists $\lambda > 0$ sufficiently large and $c > 0$ such that

$$\mathbb{E}_{\mathbb{Q}} \|X_t^{\xi} - Y_t^{\eta}\|_r^8 \leq c\,\mathrm{e}^{-8rt} \|\xi - \eta\|_r^8. \tag{2.14}$$

Rewrite the equations as follows:

$$\begin{aligned} & dX^{\xi}(t) = \{b(X_t^{\xi}) - \lambda(X^{\xi}(t) - Y^{\eta}(t))\}dt + \sigma(X_t^{\xi})d\tilde{W}(t), \\ & t > 0, X_0^{\xi} = \xi \in \mathcal{C}_r, \end{aligned} \tag{2.15}$$

and

$$dY^{\eta}(t) = b(Y_t^{\eta})dt + \sigma(Y_t^{\eta})d\tilde{W}(t),\ t > 0,\ Y_0^{\eta} = \eta \in \mathcal{C}_r. \tag{2.16}$$

Then apply the Itô formula to $e^{\lambda t}|X^{\xi}(t) - Y^{\eta}(t)|$.

Proof of the Theorem: For any $f \in C_b^1(\mathcal{X}; \mathbb{R}_+)$, by the uniqueness of weak solutions to (2.1) and (2.16), we derive that

$$\begin{aligned} \mathcal{P}_t \log f(\eta) &= \mathbb{E}_{\mathbb{Q}} \log f(Y_t^\eta) \\ &= \mathbb{E}_{\mathbb{Q}} \log f(X_t^\xi) + \mathbb{E}_{\mathbb{Q}}(\log f(Y_t^\eta) - \log f(X_t^\xi)) \\ &\le \mathbb{E}(R_\infty^{(1)} \log f(X_t^\xi)) + \|\nabla \log f\|_\infty \mathbb{E}_{\mathbb{Q}} \|X_t^\xi - Y_t^\eta\|_r \\ &\le \mathbb{E}(R_\infty^{(1)} \log R_\infty^{(1)}) + \log \mathcal{P}_t f(\xi) + \|\nabla \log f\|_\infty (\mathbb{E}_{\mathbb{Q}} \|X_t^\xi - Y_t^\eta\|_r^2)^{1/2}, \end{aligned} \tag{2.17}$$

where the last display we have applied the Hölder inequality and the Young inequality: for any random variables $f \geq 0$ with $\mathbb{E}\, f > 0$ and g.

$$\mathbb{E}(fg) \le \mathbb{E} f \log \mathbb{E} e^g + \mathbb{E}(f \log f) - \mathbb{E} f \log \mathbb{E} f.$$

Next, taking advantage of Hölder's inequality followed by (2.14) gives that

$$\mathbb{E}_{\mathbb{Q}}\|X_t^{\xi}-Y_t^{\eta}\|_r^2\le(\mathbb{E}_{\mathbb{Q}}\|X_t^{\xi}-Y_t^{\eta}\|_r^8)^{1/4}\le c^{\frac{1}{4}}\,\mathrm{e}^{-2rt}\|\xi-\eta\|_r^2,\quad \xi,\eta\in\mathcal{C}_r. \tag{2.18}$$

Subsequently, it follows from (2.18) and (**H2**) that

$$\begin{aligned}\mathbb{E}(R_\infty^{(1)}\log R_\infty^{(1)})&=\mathbb{E}_{\mathbb{Q}}\log R_\infty^{(1)}\\&=\frac{\lambda^2}{2}\mathbb{E}_{\mathbb{Q}}\int_0^\infty|\sigma^{-1}(X_t^{\xi})(X^{\xi}(t)-Y^{\eta}(t))|^2dt\\&\le\frac{c_4\,\lambda^2}{2}\int_0^\infty\mathbb{E}_{\mathbb{Q}}|X^{\xi}(t)-Y^{\eta}(t)|^2dt\\&\le\frac{c_4\,\lambda^2}{2}\int_0^\infty\mathbb{E}_{\mathbb{Q}}\|X_t^{\xi}-Y_t^{\eta}\|_r^2dt\\&\le\frac{c_4\,c^{\frac{1}{4}}\,\lambda^2}{4r}\|\xi-\eta\|_r^2\end{aligned} \tag{2.19}$$

The asymptotically strong Feller property of $\mathcal{P}_t$ follows from [Lihu Xu, 2011].

Consider the following SDEs of *neutral type* with infinite memory:

$$d\{X(t)-G(X_t)\} = b(X_t)dt+\sigma(X_t)dW(t), \quad t>0, \quad X_0=\xi\in\mathcal{C}_r, \ (3.1)$$

where b,σ and W are given as in (2.1), and $G:\mathcal{C}_r\to\mathbb{R}^d$, which is, in general, named as the neutral term of (3.1).
Besides (**H2**) and (**H3**) , we further assume that

(A1) There exists $\delta\in(0,1)$ such that $|G(\xi)-G(\eta)|\leq\delta\|\xi-\eta\|_r$ for any $\xi,\eta\in\mathcal{C}_r$;

(A2) $b\in C(\mathcal{C}_r)$ is bounded on bounded subsets of $\mathcal{C}_r$ and there exists an $L>0$ such that

$$2\langle\xi(0)-\eta(0)-(G(\xi)-G(\eta)),b(\xi)-b(\eta)\rangle\leq L\|\xi-\eta\|_r^2, \quad \xi,\eta\in\mathcal{C}_r.$$

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Theorem

Assume (**H2**)-(**H3**) and (**A1**)-(**A2**). Then the asymptotic log-Harnack inequality holds.

SPDEs case

Now, we generalize the Theorem above to the setup of path-dependent stochastic partial differential equations (SPDEs). More precisely, we are interested in the following path-dependent SPDE on the Hilbert space $\mathbb{H}$

$$dX(t) = \{AX(t) + b(X_t)\}dt + \sigma(X_t)dW(t), \ t > 0, \ X_0 = \xi, \quad (4.1)$$

where $(A, \mathscr{D}(A))$ is a densely defined closed operator on $\mathbb{H}$ generating a C_0-semigroup e^{tA}, $b : \mathcal{C}_r \to \mathbb{H}$, $\sigma : \mathcal{C}_r \to \mathscr{L}(\mathbb{H})$, $(W(t))_{t\geq 0}$ is a cylindrical Wiener process on $\mathbb{H}$ with respect to a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with the natural filtration $(\mathscr{F}_t)_{t\geq 0}$.

We assume that

(B1) $(-A, \mathscr{D}(A))$ is self-adjoint with discrete spectrum $0 < \lambda_1 \le \lambda_2 \le \cdots$ counting multiplicities such that $\lambda_i \uparrow \infty$.

(B2) For any $t > 0$, there exists an $L_0 > 0$ such that

$$|b(\xi) - b(\eta)| + \|\sigma(\xi) - \sigma(\eta)\|_{HS} \le L_0 \|\xi - \eta\|_r.$$

(B3) $\int_0^t s^{-2\alpha} \|e^{sA}\sigma(0)\|_{HS}^2 ds < \infty$ holds for some constant $\in (0, \frac{1}{2})$ and all $t > 0$.

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Under **(B1)-(B3)**, (4.1) admits a unique mild solution. That is, for any $\mathcal{F}_0$-measurable random variable $X_0 = \xi \in \mathcal{C}_r$, there exists a unique continuous adapted process $(X_t^\xi)_{t\geq 0}$ on $\mathcal{C}_r$ such that $\mathbb{P}$-a.s.

$$X^\xi(t) = e^{tA}\xi(0) + \int_0^t e^{tA} b(X_s^\xi) ds + \int_0^t e^{tA}\sigma(X_s^\xi) dW(s), \quad t > 0.$$

This can be proved by the classical Banach fixed point theorem.

The semigroup $\mathcal{P}_t$ generated by the segment process X_t corresponding to (4.1) also satisfies the asymptotic log-Harnack inequality, i.e.

Theorem

Under **(B1)-(B4)**, there exist $C_5, C_6 > 0$ such that

$$\mathcal{P}_t \log f(\eta) \le \log \mathcal{P}_t f(\xi) + C_3\|\xi - \eta\|_r^2 + C_4 \, \mathrm{e}^{-rt} \|\nabla \log f\|_\infty \|\xi - \eta\|_r \tag{4.2}$$

holds for $\xi, \eta \in \mathcal{C}_r$ and $f \in C_b^1(\mathcal{X}; \mathbb{R}_+)$. Moreover, (4.2) implies that $\mathcal{P}_t$ is asymptotically strong Feller.

Sketch of the proof

For any $\lambda > 0$, consider the following path-dependent SPDE

$$dY(t) = \{AY(t) + b(Y_t) + \lambda\sigma(Y_t)\sigma^{-1}(X_t)(X(t) - Y(t))\}dt \\ + \sigma(Y_t)dW(t), \quad t > 0 \tag{4.3}$$

with the initial value $Y_0 = \eta \in \mathcal{C}_r$. Under (**B1**)-(**B4**), (4.3) has a unique strong solution $(Y^\eta(t))_{t\geq 0}$ with the initial value $Y_0 = \eta$, where the corresponding segment process is written as $(Y_t^\eta)_{t\geq 0}$.

For any $t \geq 0$ and $\lambda > r$, set

$$h_3^{(\lambda)}(t) := \lambda\,\sigma^{-1}(X_t^{\xi})(X^{\xi}(t) - Y^{\eta}(t))$$
$$\tilde{W}(t) := W(t) + \int_0^t h_3^{(\lambda)}(s)ds.$$

Define

$$R_{\infty}^{(3)} = \exp\Big(- \int_0^{\infty} \langle h_3^{(\lambda)}(t), dW(t)\rangle - \frac{1}{2}\int_0^{\infty} |h_3^{(\lambda)}(t)|^2 dt\Big).$$

Then, due to the Girsanov theorem, $(\tilde{W}(t))_{t\geq 0}$ is a cylindrical Wiener process under the weighted probability measure $d\mathbb{Q} := R_{\infty}^{(3)} d\mathbb{P}$.

Ergodicity

As far as path-dependent SDEs are concerned, since the laws of segment processes starting from different initial values are mutually singular, the notion of small set no longer works. So the classical Harris theorem (see, [Hairer, Mattingly, Scheutzow, 2009, Theorem 4.2].) cannot be applied to investigated ergodicity for path-dependent SDEs. Recently, Hairer et al. developed a weak Harris' theorem , which can be adopted to discuss ergodicity for stochastic dynamical systems. The weak form of Harris' theorem has been applied successfully to study ergodicity for path-dependent SDEs and for Markov processes with random switching.

Consider the following path-dependent SDE on $\mathbb{R}^d$ of neutral type

$$d\{X(t) - G(X_t)\} = b(X_t)dt + \sigma(X_t)dW(t), t > 0, X_0 = \xi \in \mathcal{C}_r, \quad (5.1)$$

To investigate the exponential convergence of $P_t(\xi, \cdot)$ under the metric $\mathbb{W}_{\rho_r}$, we impose the following assumptions.

(A0) b and σ are continuous and bounded on bounded subsets of $\mathcal{C}_r$, and there exists $\in (0, 1)$ such that

$$|G(\xi) - G(\eta)| \leq \|\xi - \eta\|_r, \quad \xi, \eta \in \mathcal{C}_r.$$

(A1) There exists a constant $L_0 > 0$ such that

$$\langle \xi(0) - \eta(0) + G(\eta) - G(\xi), b(\xi) - b(\eta)\rangle^+ + \|\sigma(\xi) - \sigma(\eta)\|_{\mathrm{HS}}^2 \leq L_0\|\xi - \eta\|_r^2, \quad \xi,$$

(A2) For each $\xi \in \mathcal{C}_r$, $\sigma(\xi)$ is invertible, and $\sup_{\xi \in \mathcal{C}_r}\{\|\sigma(\xi)\| + \|\sigma(\xi)^{-1}\|\} < \infty$;

(A3) There exists a continuous function $V : \mathcal{C}_r \to \mathbb{R}_+$ with $\lim_{\|\xi\|_r \to \infty} V(\xi) = \infty$ such that

$$P_t V(\xi) \leq K e^{-\gamma t} V(\xi) + K$$

holds for some constants $K, \gamma > 0$.

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To investigate the ergodicity, we will take the Wasserstein distance induced by the distance

$$\rho_r(\xi,\eta) := 1 \wedge \|\xi-\eta\|_r, \ \ \xi,\eta \in \mathcal{C}_r. \qquad (5.2)$$

For any $\mu,\nu \in \mathscr{P}(\mathcal{C}_r)$, the collection of all probability measures on $\mathcal{C}_r$, the L^1-Warsserstein distance between μ and ν induced by ρ_r is defined by

$$\mathbb{W}_{\rho_r}(\mu,\nu) = \inf_{\pi\in\mathcal{C}(\mu,\nu)} \int_{\mathcal{C}_r\times\mathcal{C}_r} \rho_r(\xi,\eta)\pi(d\xi,d\eta), \qquad (5.3)$$

where $\mathcal{C}(\mu,\nu)$ is the set of all couplings of μ and ν; that is, $\pi \in \mathcal{C}(\mu,\nu)$ if and only if π is a probability measure on $\mathcal{C}_r \times \mathcal{C}_r$ such that $\pi(\cdot \times \mathcal{C}_r) = \mu$ and $\pi(\mathcal{C}_r \times \cdot) = \nu$.

For the Lyapunov function V in (**A3**), let

$$\rho_{r,V}(\xi,\eta) := \sqrt{\rho_r(\xi,\eta)(1+V(\xi)+V(\eta))},\ \ \xi,\eta \in \mathcal{C}_r.$$

Theorem

Assume (**A0**)-(**A3**). *Then P_t has a unique invariant probability measure π, and there exist constants $c, \lambda > 0$ such that*

$$\mathbb{W}_{\rho_{r,V}}(\mu P_t, \nu P_t) \le c\,\mathrm{e}^{-\lambda t}\mathbb{W}_{\rho_{r,V}}(\mu,\nu),\ \ \mu,\nu \in \mathscr{P}(\mathcal{C}_r),\ t \ge 0. \quad (5.4)$$

Consequently, there exists a constant $C > 0$ such that

$$\mathbb{W}_{\rho_{r,V}}(P_t(\xi,\cdot),\pi) \le C\,\mathrm{e}^{-\lambda t}\sqrt{1+V(\xi)},\ \ t \ge 0. \quad (5.5)$$

Unlike conditions (**A1**) and (**A2**) which are explicitly imposed on the coefficients, the Lyapunov condition (**A3**) is set by means of the semigroup P_t which is less explicit. In many cases, one may verify (**A3**) by using the Lyapunov condition

$$\mathscr{L}V(\xi) \leq -\lambda V(\xi) + c, \quad \xi \in \mathcal{C}_r \tag{5.6}$$

for some constants $c, \lambda > 0$.
However, $\mathscr{L}$ is not available for the present model. We now present below explicit conditions for (**A3**).

Proposition

Let $\mu_0 \in \mathscr{P}((-\infty, 0])$ *such that*

$$\delta_r(\mu_0) := \int_{-\infty}^{0} \mathrm{e}^{-2r\theta} \mu_0(d\theta) < \infty, \tag{5.7}$$

and set $\beta := \left(1 + \sqrt{1 + {}_2\delta_r(\mu_0)}\right)^2$. *Then* (**A3**) *holds for* $V(\xi) := \|\xi\|_r^2$ *provided that the following two conditions hold:*

(*i*) *For any* $\xi \in \mathcal{C}_r$, *there exist constants* ${}_1, {}_2 > 0$ *with* ${}_1 + {}_2\delta_r(\mu_0) < 1$ *such that*

$$|G(\xi)|^2 \leq {}_1|\xi(0)|^2 + {}_2 \int_{-\infty}^{0} |\xi(\theta)|^2 \mu_0(d\theta). \tag{5.8}$$

(*ii*) *There exist constants* $c_0, \lambda_1, \lambda_2 > 0$ *with* $\gamma := \lambda_1 - 2r\beta - \lambda_2\delta_r(\mu_0) > 0$ *such that*

$$2\langle \xi(0) - G(\xi), b(\xi)\rangle + \|\sigma(\xi)\|_{\mathrm{HS}}^2 \leq c_0 - \lambda_1|\xi(0)|^2 + \lambda_2 \int_{-\infty}^{0} |\xi(\theta)|^2 \mu_0(d\theta)$$

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Sketch of the Proof of the Theorem

As far as path-dependent SDEs are concerned, since the laws of segment processes starting from different initial values are mutually singular, the notion of small set no longer works. So the classical Harris theorem (see, [Hairer, Mattingly, Scheutzow, 2009, Theorem 4.2].) cannot be applied to investigated ergodicity for path-dependent SDEs. Recently, Hairer et al. developed a weak Harris' theorem , which can be adopted to discuss ergodicity for stochastic dynamical systems. The weak form of Harris' theorem has been applied successfully to study ergodicity for path-dependent SDEs and for Markov processes with random switching.
We also use weak Harris' theorem, see, e.g., [Hairer, Mattingly, Scheutzow, 2009, Theorem 4.8]. To present Harris' theorem smoothly, let's recall some notions.

Definition

Let $\mathbb{X}$ be a Polish space, and $(P_t)_{t\geq 0}$ a Markov semigroup with transition kernel $P_t(\xi,\cdot)$ on $\mathbb{X}$.

(1) *A continuous function $V:\mathbb{X}\to\mathbb{R}_+$ is called a Lyapunov function for $(P_t)_{t\geq 0}$, if there exist constants $\gamma, K>0$ such that*

$$P_tV(\xi):=\int_{\mathbb{X}}V(\eta)P_t(\xi,d\eta)\leq K\,\mathrm{e}^{-\gamma t}V(\xi)+K,\qquad \xi\in\mathbb{X},\quad t\geq 0. \tag{5.11}$$

(2) *A function $\rho:\mathbb{X}\times\mathbb{X}\to[0,1]$ is said to be distance-like if it is symmetric, lower semi-continuous, and $\rho(\xi,\eta)=0$ if and only if $\xi=\eta$.*

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Definition

(3) *A set $A \subset \mathbb{X}$ is said to be ρ-small for P_t, if there exists $\varepsilon \in (0,1)$ such that*

$$\mathbb{W}_\rho(P_t(\xi,\cdot), P_t(\eta,\cdot)) \leq 1-\varepsilon, \quad \xi,\eta \in A,$$

where $\mathbb{W}_\rho$ is defined as in (5.3) for $(\mathbb{X},\rho)$ replacing $(\mathcal{C}_r,\rho_r)$.

(4) *ρ is said to be contractive for P_t, if there exists $\varepsilon \in (0,1)$ such that*

$$\mathbb{W}_\rho(P_t(\xi,\cdot), P_t(\eta,\cdot)) \leq \varepsilon\,\rho(\xi,\eta), \quad \xi,\eta \in \mathbb{X} \ \text{ with } \rho(\xi,\eta) < 1.$$

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Theorem

Let ρ be a distance-like function on $\mathbb{X} \times \mathbb{X}$, and V a Lyapunov function such that (5.11) *holds for some constants $\gamma, K > 0$. If there exists a constant $t^* > 0$ such that $\{V \leq 4K\}$ is ρ-small and ρ is contractive for P_{t^*}, then there exists a constant $t > 0$ such that*

$$\mathbb{W}_{\rho_V}(\mu P_t, \nu P_t) \leq \frac{1}{2}\mathbb{W}_{\rho_V}(\mu, \nu), \quad \forall\ \mu, \nu \in \mathscr{P}(\mathbb{X}),$$

where $\rho_V(\xi, \eta) := \sqrt{\rho(\xi, \eta)(1 + V(\xi) + V(\eta))}, \xi, \eta \in \mathbb{X}.$

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